A NOTE ON INDUCED 4-DIMENSIONAL LORENTZ TRANSFORMATION

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ABSTRACT. In the present note a set of six typical transformation schemes has been so framed that each of them leads to a representation of the 4-dimensional Lorentz transformation both proper and improper. It is shown that corresponding to a particular transformation there exists a mixed tensor of 2nd rank, which while undergoing the transformation can induce a 4-dimensional Lorentz transformation to a 4-vector associated with it yielding the connecting relations between the respective transformation coefficients. Further, under each of these transformation schemes one can set up a system of Dirac equations and construct an electromagnetic tensor whence the set of Maxwell's Equations can be formulated.

In an R_4 characterized by a set of general reversible transformation equations from coordinates x^r to x'^r with 16 covariant transformation coefficients $\partial x^r/\partial x'^p$ one can define the 6 special transformation schemes by making suitable use of the following six elementary anti-symmetric covariant tensors

$$\begin{aligned} &C_{pq} \text{ with non-vanishing components } C_{01} = C_{23} = 1,\ C_{10} = C_{32} = -1,\\ &D_{pq} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad C_{02} = C_{13} = 1,\ C_{20} = C_{31} = -1, \qquad \ldots \quad (1.1)\\ &E_{pq} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad C_{03} = C_{12} = 1,\ C_{30} = C_{21} = -1,\end{aligned}$$

and their conjugates

$$\begin{aligned} &\bar{C}_{pq} \text{ with nonvanishing components } \bar{C}_{01} = \bar{C}_{32} = -1,\ \bar{C}_{10} = \bar{C}_{23} = 1,\\ &\bar{D}_{pq} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \bar{C}_{02} = \bar{C}_{31} = -1,\ \bar{C}_{20} = \bar{C}_{13} = 1, \qquad \ldots \quad (1.2)\\ &\bar{E}_{pq} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \bar{C}_{03} = \bar{C}_{21} = -1,\ \bar{C}_{30} = \bar{C}_{12} = 1.\end{aligned}$$

To denote the contravariant tensor associated to each of the above we write the indices as superscripts.

Let us consider the first one which we call the $(C-\bar{D})$ transformation under which we postulate that C_{pq}, the primary tensor, remains invariant and $\bar{D}_{pq}$ the secondary, goes over into $\lambda \bar{D}_{pq}$, where $\lambda = \pm 1$. The conditions which the transformation coefficients $\partial x^r/\partial x'^p$ denoted by (r_p), must satisfy are obtained as follows :

$$\begin{bmatrix} \binom{1}{1}-\binom{0}{1} & \binom{3}{1}-\binom{2}{1} \\ -\binom{1}{0} & \binom{0}{0}-\binom{3}{0} & \binom{2}{0} \\ \binom{1}{3}-\binom{0}{3} & \binom{3}{3}-\binom{2}{3} \\ -\binom{1}{2} & \binom{0}{2}-\binom{3}{2} & \binom{2}{2} \end{bmatrix} \begin{bmatrix} \binom{0}{0} & \binom{0}{1} & \binom{0}{2} & \binom{0}{3} \\ \binom{1}{0} & \binom{1}{1} & \binom{1}{2} & \binom{1}{3} \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.3)$$

$$\begin{bmatrix} \binom{2}{2}-\binom{3}{2} & \binom{0}{2} & \binom{1}{2} \\ -\binom{2}{3} & \binom{3}{3} & \binom{0}{3}-\binom{1}{3} \\ -\binom{2}{0} & \binom{3}{0} & \binom{0}{0}-\binom{1}{0} \\ \binom{2}{1}-\binom{3}{1}- & \binom{0}{1} & \binom{1}{1} \end{bmatrix} \begin{bmatrix} \binom{0}{0} & \binom{0}{1} & \binom{0}{2} & \binom{0}{3} \\ \binom{1}{0} & \binom{1}{1} & \binom{1}{2} & \binom{1}{3} \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

The above are equivalent to the conditions

$$\begin{aligned} &\binom{1}{1} = \lambda\binom{2}{2}, \quad \binom{0}{1} = \lambda\binom{3}{2}, \\ &\binom{1}{0} = \lambda\binom{2}{3}, \quad \binom{0}{0} = \lambda\binom{3}{3} \\ &\binom{1}{3} = -\lambda\binom{2}{0} \quad \binom{0}{3} = -\lambda\binom{3}{0} \\ &\binom{1}{2} = -\lambda\binom{2}{1} \quad \binom{0}{2} = -\lambda\binom{3}{1} \\ &\binom{01}{01} + \binom{23}{01} = 1, \quad \binom{01}{02} + \binom{23}{02} = 0, \end{aligned} \quad \ldots (1.4)$$

where the symbol $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ denotes $\binom{p}{r}\binom{q}{s} - \binom{p}{s}\binom{q}{r}$.

Elsewhere (Ghosh, 1965) this special $(C-\overline{D})$ trasformation has been termed 'Unimodular tensor transformation' and has been discussed at length giving the representation of an 'induced' 4-dimensional Lorentz transformation and the derivation of the corresponding Dirac equations.

Here, we shall construct the electromagnetic mixed tensor of the second rank in the $(C-\bar{D})$ field. Referring to formulae (6.2, 3, 4) of the earlier paper (Ghosh, 1965) we notice that if there exists a mixed tensor F_q^p in the $(C-\bar{D})$ field satisfying the structural relations

$$\begin{aligned} F_0^{\,0} &= -F_1^{\,1} = -F_2^{\,2} = F_3^{\,3}, \\ F_2^{\,1} &= F_0^{\,3} = -F_1^{\,2} = -F_3^{\,0}, \\ F_2^{\,0} &= -F_1^{\,3},\ F_0^{\,2} = -F_3^{\,1}, \\ F_1^{\,0} &= F_2^{\,3},\ F_0^{\,1} = F_3^{\,2}, \end{aligned} \qquad \dots \quad (1.5)$$

with the 6 mutually independent components $F_0^{\,0}$, $F_2^{\,1}$, $F_2^{\,0}$, $F_0^{\,2}$, $F_1^{\,0}$, $F_0^{\,1}$ then we can correlate an antisymmetric tensor $E_k^{\,l}$ by means of the equation

$$E_k^{\,l} = \tfrac{1}{4}\, T_{kp}^{\;\;r}[T_r^{\,lm}F_m^{\,p} - T_m^{\,lp}F_r^{\,m}],\ (k, l = 0, 1, 2, 3) \qquad \dots \quad (1.6)$$

where T's are the connecting tensors characteristic for the $(C-\bar{D})$ transformation. It is easy to see that $F_q^{\,p}$ taken in the bilinear form $A^pB_q+B^pA_q$ satisfying the structural relation

$$F_q^{\,p}C^{qr}C'_{ps} = -F_s^{\,r}, \qquad \dots \quad (1.7)$$

supplemented by the conditions $F_0^{\,0} = -F_2^{\,2}$, $F_2^{\,1} = -F_1^{\,2}$, $F_1^{\,0} = F_2^{\,3}$, $F_0^{\,1} = F_3^{\,2}$, is the desired mixed tensor in the $(C-\bar{D})$ field.

We shall next consider the $(C-E)$ transformation scheme under which the tensor C_{pq} remains invariant and the tensor E_{pq} goes over into λE_{pq} where $\lambda = \pm 1$. The conditions which the transformation coefficients $\binom{r}{p}$ must satisfy are then given as

$$\begin{aligned} \binom{1}{1} &= \lambda\binom{3}{3}, & \binom{0}{1} &= -\lambda\binom{2}{3}, \\ \binom{1}{0} &= -\lambda\binom{3}{2}, & \binom{0}{0} &= \lambda\binom{2}{2}, \\ \binom{1}{3} &= -\lambda\binom{3}{1}, & \binom{0}{3} &= \lambda\binom{2}{1}, \\ \binom{1}{2} &= \lambda\binom{3}{0}, & \binom{0}{2} &= -\lambda\binom{2}{0}, \\ \binom{01}{01}+\binom{23}{01} &= 1, & \binom{01}{03}+\binom{23}{03} &= 0. \end{aligned} \qquad \dots \quad (2.1)$$

It may be noted here that the contravariant and covariant transformation coefficients are related as in $(C-\bar{D})$ transformation and the rule of raising and lowering of indices remain unchanged. Consider now the mixed tensor $M_r{}^p$ expressed in the bilinear form $A_pB^r - B_pA^r$ satisfying the structural relation

$$M_r{}^p\, C^{rq}C_{ps} = M_s{}^q \qquad .. \quad (2.2)$$

with 4 mutually independent components defined in terms of 4 quantities h_k by means of the equations

$$\begin{aligned} &M_0{}^0 = M_1{}^1 = -M_2{}^2 = -M_3{}^3 = -h_2, \\ &M_2{}^0 = M_1{}^3 = M_0{}^2 = M_3{}^1 = h_1, \\ &M_2{}^1 = -M_0{}^3 = h_0+h_3, \quad M_1{}^2 = -M_3{}^0 = -h_0+h_3, \\ &M_1{}^0 = M_0{}^1 = M_3{}^2 = M_2{}^3 = 0. \end{aligned} \qquad .. \quad (2.3)$$

Introducing a set of connecting tensors $T_q{}^kp (k = 0, 1, 2, 3)$ defined by the nonvanishing components

$$\begin{aligned} &T^{00}{}_3 = 1, \quad T^0{}_2{}^1 = 1, \quad T^0{}_1{}^2 = -1, \quad T^0{}_0{}^3 = -1. \\ &T^1{}_2{}^0 = 1, \quad T^1{}_3{}^1 = 1, \quad T^1{}_0{}^2 = 1, \quad T^1{}_1{}^3 = 1, \\ &T^2{}_0{}^0 = -1, \quad T^2{}_1{}^1 = -1, \quad T^2{}_2{}^2 = 1, \quad T^2{}_3{}^3 = 1, \\ &T^3{}_3{}^0 = -1, \quad T^3{}_2{}^1 = 1, \quad T^3{}_1{}^2 = 1, \quad T^3{}_0{}^3 = -1, \end{aligned} \qquad .. \quad (2.4$$

the above can be expressed as

$$M_r{}^p = T_r{}^kp\; h_k \quad (k = 0, 1, 2, 3) \qquad .. \quad (2.5)$$

which being inverted gives

$$h_k = \tfrac{1}{4}\, T_{kp}{}^r M^p{}_r, \qquad .. \quad (2.6)$$

where
$$T^r{}_{kp} = g_{kl}T_p{}^{lr}.$$

g_{kl} denoting the tensor with non vanishing components

$$g_{00} = -1, \quad g_{11} = g_{22} = g_{33} = 1.$$

The set of 4 connecting tensors (2.4) is characteristic for the $(C-E)$ transformation. These in conjunction with the mixed tensor (2.3) lead to the representation of Lorentz transformation and to the derivation of Dirac equation. Under $(C-E)$ transformation scheme one can verify that the tensor $F_q{}^p$ taken in the linear form $A^pB_q+B^pA_q$ supplemented by the conditions $F_0{}^1 = -F_2{}^3$, $F_1{}^0 = -F_3{}^2$ $F_0{}^0 = F_2{}^2$, $F^0{}_2 = -F_0{}^2$, having 6 mutually independent components $F_0{}^0$, $F_2{}^0$, $F_2{}^1$, $F_1{}^2$, $F_1{}^0$, $F_0{}^1$ serves as electromagnetic tensor.

With D_{pq} as primary tensor we next consider the $(D-\bar{C})$ transformation scheme under which D_{pq} remains invariant and $\bar{C}_{pq}$ goes over into $\lambda\bar{C}_{pq}$ where

$\lambda = \pm 1$. The conditions which the transformation coefficients $\binom{r}{p}$ must satisfy are then given by

$$\begin{aligned}
&\binom{2}{2} = \lambda\binom{1}{1}, \quad \binom{3}{2} = -\lambda\binom{0}{1},\\
&\binom{2}{3} = -\lambda\binom{1}{0}, \quad \binom{3}{3} = \lambda\binom{0}{0},\\
&\binom{2}{0} = \lambda\binom{1}{3}, \quad \binom{3}{0} = -\lambda\binom{0}{3},\\
&\binom{2}{1} = -\lambda\binom{1}{2}, \quad \binom{3}{1} = \lambda\binom{0}{2},\\
&\binom{02}{13}+\binom{13}{13} = 1, \quad \binom{02}{01}+\binom{13}{01} = 0.
\end{aligned} \tag{3.1}$$

The contravariant and covariant transformation coefficients $\left\{{q \atop s}\right\}$, $\binom{r}{p}$ are now connected by the equation

$$\left\{{q \atop s}\right\} = D_{rs}D^{pq}\binom{r}{p}. \qquad \ldots \quad (3.2)$$

The raising and lowering of indices may be performed under the scheme

$$A^0 = A_2, \quad A^2 = -A_0, \quad A^1 = A_3, \quad A^3 = -A_1. \qquad \ldots (3.3)$$

We note here the relations

$$A^pA_p = 0, \quad A^pB_p+B^pA_p = 0. \qquad \ldots \quad (3.4)$$

Let us now construct a mixed tensor $N^p{}_q$ by means of a pair of tensors A^p, B_q taken in the bilinear form $A^pB_q-B^pA_q$ satisfying the structural equation

$$N_s{}^q = N_p{}^rD_{rs}D^{pq} \qquad \ldots \quad (3.5)$$

having 4 mutually independent components expressed in terms of 4 quantities h_k by means of the equations

$$\begin{aligned}
&N_0{}^0 = N_2{}^2 = -N_1{}^1 = -N_3{}^3 = -h_2,\\
&N_2{}^1 = -N_3{}^0 = N_1{}^2 = -N_0{}^3 = h_1,\\
&N^0{}_1 = N_2{}^3 = h_0+h_3, \quad N_0{}^1 = N_3{}^2 = -h_0+h_3,\\
&N^0{}_2 = N_0{}^2 = N_3{}^1 = N_1{}^3 = 0.
\end{aligned} \qquad \ldots \quad (3.6)$$

Introducing the 4 connecting tensors

$T^{0}{}_{q}{}^{p}$ with non vanishing components $\left|{0 \atop 1}\right. = \left|{3 \atop 2}\right. = -\left|{1 \atop 0}\right. = -\left|{2 \atop 3}\right. = 1,$

$T^{1}{}_{q}{}^{p}$ . . . . $\left|{0 \atop 3}\right. = \left|{3 \atop 0}\right. = -\left|{1 \atop 2}\right. = -\left|{2 \atop 1}\right. = -1,$

... (3.7)

$T^{2}{}_{q}{}^{p}$ . . . . $\left|{0 \atop 0}\right. = \left|{2 \atop 2}\right. = -\left|{1 \atop 1}\right. = -\left|{3 \atop 3}\right. = -1,$

$T^{3}{}_{q}{}^{p}$ . . . . $\left|{0 \atop 1}\right. = \left|{2 \atop 3}\right. = \left|{1 \atop 0}\right. = \left|{3 \atop 2}\right. = 1.$

we can express the above as

$$N_{q}{}^{p} = T^{k}{}_{q}{}^{p} h_{k}. \qquad \text{... (3.8)}$$

Using the above connecting tensors, characteristic for the (D—C) transformation one can obtain further results in this connection.

It appears from (3.6) that we can connect h_k in a different way with the N's so that a new transformation (D—E) is obtained. We take

$$\begin{aligned} &N_{0}{}^{0} = N_{2}{}^{2} = -N_{1}{}^{1} = -N_{3}{}^{3} = -h_{2}, \\ &N_{1}{}^{0} = N_{2}{}^{3} = N_{0}{}^{1} = N_{3}{}^{2} = h_{1}, \\ &N_{3}{}^{0} = -N_{2}{}^{1} = h_{0}+h_{3}, \;\; N_{0}{}^{3} = -N_{1}{}^{2} = -h_{0}+h_{3}, \\ &N_{2}{}^{0} = N_{0}{}^{2} = N_{3}{}^{1} = N_{1}{}^{3} = 0. \end{aligned} \qquad \text{... (3.9)}$$

Introducing the characteristic connecting tensors

$T^{0}{}_{q}{}^{p}$ with nonvanishing components $\left|{0 \atop 3}\right. = \left|{2 \atop 1}\right. = -\left|{3 \atop 0}\right. = -\left|{1 \atop 2}\right. = 1.$

$T^{1}{}_{q}{}^{p}$ . . . . $\left|{0 \atop 1}\right. = \left|{1 \atop 0}\right. = \left|{2 \atop 3}\right. = \left|{3 \atop 2}\right. = 1,$

... (3.10)

$T^{2}{}_{q}{}^{p}$ . . . . $\left|{0 \atop 0}\right. = \left|{2 \atop 2}\right. = -\left|{1 \atop 1}\right. = -\left|{3 \atop 3}\right. = -1,$

$T^{3}{}_{q}{}^{p}$ . . . . $\left|{0 \atop 3}\right. = \left|{3 \atop 0}\right. = -\left|{1 \atop 2}\right. = -\left|{2 \atop 1}\right. = 1,$

we can express the above as

$$N_{q}{}^{p} = T^{k}{}_{q}{}^{p} h_{k}. \qquad \dots \quad (3.11)$$

Under $(D-\bar{E})$ transformation the tensor D_{pq} remains invariant and $\bar{E}_{pq}$ goes over into $\lambda\bar{E}_{pq}$. The conditions which the transformation coefficients must satisfy are given by

$$\begin{aligned}
&\binom{2}{2} = \lambda\binom{3}{3} \qquad && \binom{0}{2} = -\lambda\binom{1}{3}, \\
&\binom{2}{3} = -\lambda\binom{3}{2} && \binom{0}{3} = \lambda\binom{1}{2} \\
&\binom{2}{0} = -\lambda\binom{3}{1} && \binom{0}{0} = \lambda\binom{1}{1}, \\
&\binom{2}{1} = \lambda\binom{3}{0} && \binom{0}{1} = -\lambda\binom{1}{0}, \\
&\binom{02}{13} + \binom{13}{13} = 1, && \binom{02}{03} + \binom{13}{03} = 0.
\end{aligned} \qquad \dots \quad (3.12)$$

It may be noted that under $(D-E)$ transformation the formulae (3.2), (3.3) and (3.4) hold good retaining all characteristic properties.

With E_{pq} as primary tensor we can frame two transformation schemes by taking either C_{pq} as secondary or $\bar{D}_{pq}$ as secondary. In the former the tensor E_{pq} remains invariant and C_{pq} goes over into λC_{pq}. The conditions which the transformation coefficients $({}^{r}{}_{p})$ must satisfy are given as

$$\begin{aligned}
&\binom{3}{3} = \lambda\binom{1}{1} \qquad && \binom{2}{3} = -\lambda\binom{0}{1}, \\
&\binom{3}{2} = -\lambda\binom{1}{0} && \binom{2}{2} = \lambda\binom{0}{0} \\
&\binom{3}{1} = -\lambda\binom{1}{3} && \binom{2}{1} = \lambda\binom{0}{3}, \\
&\binom{3}{0} = \lambda\binom{1}{2} && \binom{2}{0} = -\lambda\binom{0}{2} \\
&\binom{03}{03} + \binom{12}{03} = 1, && \binom{03}{01} + \binom{12}{01} = 0.
\end{aligned} \qquad (4.1)$$

In the latter E_{pq} remains invariant and $\overline{D}_{pq}$ goes over into λD_{pq} where $\lambda = \pm 1$. The conditions which the transformation coefficients must satisfy are given as follows :

$$
\begin{aligned}
&\binom{3}{3} = \lambda \binom{2}{2}, \qquad \binom{1}{3} = \lambda \binom{0}{2},\\
&\binom{3}{2} = -\lambda \binom{2}{3}, \qquad \binom{1}{2} = -\lambda \binom{0}{3},\\
&\binom{3}{1} = \lambda \binom{2}{0}, \qquad \binom{1}{1} = \lambda \binom{0}{0},\\
&\binom{3}{0} = -\lambda \binom{2}{1}. \qquad \binom{1}{0} = -\lambda \binom{0}{1},\\
&\binom{03}{03} + \binom{12}{03} = 1, \quad \binom{03}{02} + \binom{12}{02} = 0.
\end{aligned} \qquad \dots (4.2)
$$

It may be noted in both the transformation schemes that the contravariant and the covariant transformation coefficients are connected by the equation

$$\begin{Bmatrix} q \\ s \end{Bmatrix} = E_{rs}E^{pq}\binom{r}{p} \qquad \dots (4.3)$$

Raising and lowering of indices may be performed in both according to the rule

$$A_q E^{pq} = A^p, \qquad A^p E_{pq} = A_q, \qquad \dots (4.4)$$

so that $\quad A^0 = A_3, \quad A^3 = -A_0, \quad A^1 = A_2, \quad A^2 = -A_1$

and the relations $A^p A_p = 0$, $A^p B_p + B^p A_p = 0$ hold good in both.

The characteristic connecting tensors with regard to $(E-C)$ transformation are now constructed with the nonvanishing components

$$
\begin{aligned}
&T^0{}_1{}^0 = 1, \; T^0{}_0{}^1 = -1, \; T^0{}_3{}^2 = 1, \; T^0{}_2{}^3 = -1,\\
&T^1{}_2{}^0 = 1, \; T^1{}_3{}^1 = -1, \; T^1{}_0{}^2 = 1. \; T^1{}_1{}^3 = -1,\\
&T^2{}_0{}^0 = -1, \; T^2{}_1{}^1 = 1, \; T^2{}_2{}^2 = 1, \; T^2{}_3{}^3 = -1\\
&T^3{}_1{}^0 = 1, \; T^3{}_0{}^1 = 1, \quad T^3{}_3{}^2 = 1, \; T^3{}_2{}^3 = 1,
\end{aligned} \qquad \dots (4.5)
$$

while in the $(E-\bar{D})$ transformation the characteristic connecting tensors are formed by the nonvanishing components

$$
\begin{aligned}
&T^0{}_2{}^0 = -1, \; T^0{}_3{}^1 = 1, \quad T^0{}_0{}^2 = 1, \quad T^0{}_1{}^3 = -1,\\
&T^1{}_1{}^0 = 1, \quad T^1{}_0{}^1 = 1, \quad T^1{}_3{}^2 = 1, \quad T^1{}_2{}^3 = 1,\\
&T^2{}_0{}^0 = -1, \; T^2{}_1{}^1 = 1, \quad T^2{}_2{}^2 = 1, \quad T^2{}_3{}^3 = -1,\\
&T^3{}_2{}^0 = 1, \quad T^3{}_3{}^1 = -1, \; T^3{}_0{}^2 = 1, \quad T^3{}_1{}^3 = -1.
\end{aligned} \qquad \dots (4.6)
$$

Using the standard formula in my earlier paper (Ghosh, 1965) one can get a representation of the induced 4-dimensional Lorentz transformation corresponding to each of the transformation schemes $(E-C)$ and $(E\text{-}\breve{D})$. Proceeding as before further results in this connection will follow.

With the conjugate tensors $\breve{C}_{pq}$, $\breve{D}_{pq}$, $\breve{E}_{pq}$ a set of 6 transformation schemes can be formed having similar properties.

REFERENCE

Ghosh, N. N., 1965, *Indian J. Phys.*, **39**, 435.